A MATHEMATICAL FORMULATION OF FOREST INVENTORY

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ABSTRACT

This paper considers forest inventory from mathematical points of view. Several text-book results have been proved mathematically. Also, a possible mathematical definition of time of felling of a tree has been suggested. Finally, using the results of this investigation, the mathematical model presented by Choudhury and Chowdhury (1983) has been examined and an alternative model has been proposed.

INTRODUCTION

Forest inventory in Bangladesh was studied by Anon (1960), Slavicky (1978), Choudhury and Chowdhury (1983) and others. The literature concludes that Choudhury and Chowdhury (1983) gave a mathematical formulation for volume-age relationship.

This paper considers mathematical formulation of forest inventory. Firstly, definitions of some mathematical notations used in this paper are given. Subsequently, some fundamental terminology of forest inventory are treated. Attempts have been made to prove some text-book results from the mathematical point of view and to workout a mathematical definition of the time of felling of a tree. Finally, some mathematical aspects of the model considered by Choudhury and Chowdhury (1983) have been discussed and an alternative model has been proposed.

MATHEMATICAL PRELIMINARIES

Before penetrating into the details of the subject, some mathematical notations are defined as follows :

Def. 1 : Let f(t) be a continuous function defined on the interval $[0, \infty)$. The function f(t) is called increasing (or decreasing) in $t \in [0, \infty)$ if and only if $f(t) \geq f(x)$ (or $f(t) \leq f(x)$) for all $t \geq x \geq 0$.

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If f(t) is differentiable over $(0,\infty)$ then it can be shown that (see, for example, Flett 1966, p–142),

$$f(t) \text{ is } \left\{\begin{matrix}\text{increasing}\\ \text{decreasing}\end{matrix}\right\} \text{ in } t \text{ if and only if } f'(t) \left\{\begin{matrix}\geq\\ \leq\end{matrix}\right\} 0;$$

also. if strict inequality signs held, f(t) is called strictly increasing and strictly decreasing respectively.

Def. 2 : Let f(t) be a function continuous on $[0,\infty)$. f(t) is called a convex (or concave) function if and only if for any $t, x E[0,\infty)$ and for any $\lambda E(0,1)$, $f(\lambda t+(1-\lambda\ x \leq \lambda f(t)+(1-\lambda) f(x)$(or $f(\lambda t+(1-\lambda)x) \geq \lambda f(t)+(1-\lambda) f(x)$).

It can be shown that (Flett 1966, p–180), if f(t) is twice differentiable on $(0,\infty)$, then

$$f(t) \text{ is a } \left\{\begin{matrix}\text{convex}\\ \text{concave}\end{matrix}\right\} \text{ function of } t \text{ if and only if } f''(t) \left\{\begin{matrix}\geq\\ \leq\end{matrix}\right\} 0;$$

also, f(t) is called strictly convex and strictly concave respectively if and only if strict inequality signs hold in the above relations.

Combining the conditions of Defs. 1 and 2, it is seen that f(t) is convex (concave) if and only if the function $f'(t)$ (which represents the instantaneous rate of change of the function f(t) is increasing (decreasing).

Def. 3 : Let f(t) be a function continuous on $[0,\infty)$. f(t) is said to have the absolute maximum (absolute minimum) value, denoted by f_{max} (f_{min}) if and only if $f_{max} \geq f(t)$ ($f_{min} \leq f(t)$) for all $t E[0,\infty)$.

If the function f(t) is differentiable on $(0.\infty)$ and if f(t) attains its maximum (minimum) value at some interior point of the interval $[0.\infty)$, then it can be shown that (Fleet 1966, p–170) that the point t_0 at which f(t) attains its maximum (minimum) value is the solution of the equation $f'(t)=0$.

Remark 1 : The above Defs. 1–3 can also be extended to any subset (bounded or unbounded) of the positive real axis on which f(t) is defined.

Now, let $f:[0,\infty)\rightarrow[0,\infty)$ be a continuous function which is differentiable on $(0,\infty)$ and is such that

$f(0)=0$, (1)

f(t) is strictly increasing in t on $[0,\infty)$ (2)

$f'(t)$ is strictly increasing on $(O, T']$ and strictly decreasing on $[T',\infty)$, (3)

$f(t)/t$ is strictly increasing on $(O, T]$ and and strictly decreasing on $[T,\infty)$. (4)

It is to note here that the conditions (3) and (4) require that $t=T'$ and $t=T$ are respectively the points at which $f'(t)$ and $f(t)/t$ attain their (absolute) maxima.

Now, since

$$\frac{d}{dt}[f(t)/t]=[tf'(t)-f(t)]/t^2, \quad (5)$$

and since $f(t)/t$ has the maximum at $t=T$, we have,

$$f'(T)=f(T)/T. \quad (6)$$

The relation (6) shows that, at the point $t=T$ where the curve $f(t)/t$ attains its maximum, the two curves $y=f'(t)$ and $y=f(t)/t$ (plotted against a common t-axis) intersect each other.

Also, from (5), it is found that

$$\frac{d}{dt}\left[f(t)/t\right] \text{is} \left\{\begin{matrix}>\\ <\end{matrix}\right\} 0 \text{ according as } f'(t) \left\{\begin{matrix}>\\ <\end{matrix}\right\} f(t)/t. \quad (7)$$

Therefore, combining all these results and referring to Defs. 2 and 3, and also to condition (4), we have the following

Theorem 1 : Let $f(t)$ be a function continuous on $[0, \infty)$ and differentiable on $(0, \infty)$ such that the conditions (1)–(4) hold true. Then

$$f'(t) > f(t)/t \text{ on } (0, T), \quad (8)$$

$$f'(t) < f(t)/t \text{ on } (T, \infty). \quad (9)$$

Also, $f(t)/t$ has the maximum at $t=T$ satisfying the condition (6) such that the curves $y=f'(t)$ and $y=f(t)/t$ (plotted against a common t-axis) intersect each other at $t=T$. Finally, $T>T'$, that is, the point at which the curve $y=f'(t)$ attains its maximum is to the left of the point at which the curve $y=f(t)/t$ attains its maximum.

Fig. 1 shows the qualitative behaviour of the functions $f'(t)$ and $f(t)/t$, plotted against a common t-axis(where the value 0 is assigned arbitrarily at $t=0$ for the latter function).

MATHEMATICAL FORMULATION OF FOREST INVENTORY

In this section, attempts will be made to apply the concepts and results of the previous section to forest inventory. To this end, some useful definitions of forest inventory need to be recalled.

Def. 4 : The Current Annual Increment (CAI) of a tree at age t (in years) is its increase in volume during the next year.

However, for practical purpose, the volume increment is recorded for a fixed interval of time (generally, 5 or 10 years), and the average increment during this time interval, called the Periodic Annual Increment (PAI), is regarded as the corresponding value of CAI for the time interval under consideration (Avery 1967 ; Khattak 1970).

Def. 5 : The Mean Annual Increment (MAI) of a tree at age t (in years) is the average increment per year, and is given simply by the ratio of the volume at age t to the age.

Let $V(t)$ be the total volume of a tree at age t (in years). Then,

$$\text{CAI at age } t=V'(t), \quad (10)$$

$$\text{MAI at age } t=V(t)/t. \quad (11)$$

From the definition of $V(t)$, it is clear that $V(t)$ satisfies the following two conditions.

$$V(0)=0, \quad (12)$$

$V(t)$ is strictly increasing in t on $[0, \infty)$. (13)

The conditions (12) simply states the trivial fact that, at age 0, i. e., at the beginning of the plantation, the volume is 0 ; the condition (13) states that, volume increases with increasing age. Furthermore, the continuity of $V(t)$ requires that the condition (12) is to be supplemented by the following one :

$$\lim_{t \to 0^+} V(t)=V(0)=0, \quad (14)$$

(where the symbol $\lim_{t \to 0^+} V(t) \equiv \lim_{\substack{t \to 0 \\ t>0}} V(t)$ stands for the fact that the limiting value of $V(t)$ is to be calculated as t tends to 0 along the positive t-axis, i. e., through values of $t>0$ (Flett 1966, p–108).

The conditions (12) and (13) correspond respectively to (1) and (2). The results corresponding to the conditions (3) and (4) are :

$V(t)$ is convexly increasing on $[0, T']$ and is concavely increasing on $[T', \infty)$ (15)

$V(t)/t$ is strictly increasing on $(0, T]$ and is strictly decreasing on $[T, \infty)$, (16)
which are found experimentally. Hence, the results parallel to those of Theorem 1 may be expressed as

Theorem 2 : The CAI and MAI curves interest each other at $t=T$ at which the MAI curve attains its maximum value. Also,

(i) $V'(t) > V(t)/t$ on $(0, T)$,

(ii) $V'(t) < V(t)/t$ on (T, ∞) :

that is, the CAI curve is above the MAI curve ovei the interval (0, T) but it is below the MAI curve over (T, ∞), and hence, the CAI curve attains its maximum value at a point to the left of T.

Remark 2 : It is to note here that, the conditions (1)–(4), or alternatively, conditions (12)–(16) completely characterize the qualitative behaviour of the volume-age relationship, i. e., any function representing the volume-age relationship must satisfy either set of conditions. However, the conditions (12)—(16) are closer to the facts found experimentally.

In forest inventory, another term of interest is the time of felling of a tree defined as follows :

Def. 6 : The time of felling of a tree is the time at which it is optimal (in view of volume output) to replace the tree with a new plantation.

Mathematically, $t=t_{fell}$ would be called the time of felling of a particular tree if

$$V(2t_{fell}) < 2V(t_{fell}). \qquad (17)$$

To understand the meaning of the above inequality, it is sufficient to realise that the term $V(2t_{fell})$ is the volume output when the tree is allowed to survive upto time $2t_{fell}$, while the term on the ı.h.s. of the condition (17) is the volume output obtainable from the optimal policy of employing the new plantation scheme at time t_{fell}.

The following theorem may be proved.

Theorem 3 : $t_{fell}=T$, that is, the time of felling of a tree is the time at which the CAI and MAI curves meet.

Proof : Since by condition (16), $V(t)/t$ is strictly decreasing in t on the interval (T, ∞), from Def. 1,

$$V(T+x)/(T+x) < V(T)/T \text{ for all } x > 0,$$

and so in particular, with $T=x$, the following inequality results :

$$V(2T)/2T < V(T)/T,$$

so that T satisfies the condition (17). Hence, $t_{fell}=T$. ▱

APPLICATION

A mathematical treatment of forest inventory in Bangladesh has been done by Choudhury and Chowdhury (1983). Therefore, it might be of some interest to apply the above results to the model considered by them.

The following model was considered by Choudhury and Chowdhury (1983) in connection with the volume-age relationship for particular types of tree species :

$$\ln V(t)=a+b/t.$$

Rewriting it in the equivalent form $V(t)=e^{a+b/t}$, it is noted that $b<0$ in order that the condition (14) is satisfied. Therefore, the following model may be considered :

$$\ln V(t)=a-b/t, \; b>0. \qquad (18)$$

Also, since

$$V'(t)/V(t)=b/t^2 \text{ for all } t>0,$$

it follows that the condition (13) also holds true ; moreover, b must satisfy the following condition in order that the relationship (6) is satisfied :

$$b=T. \qquad (19)$$

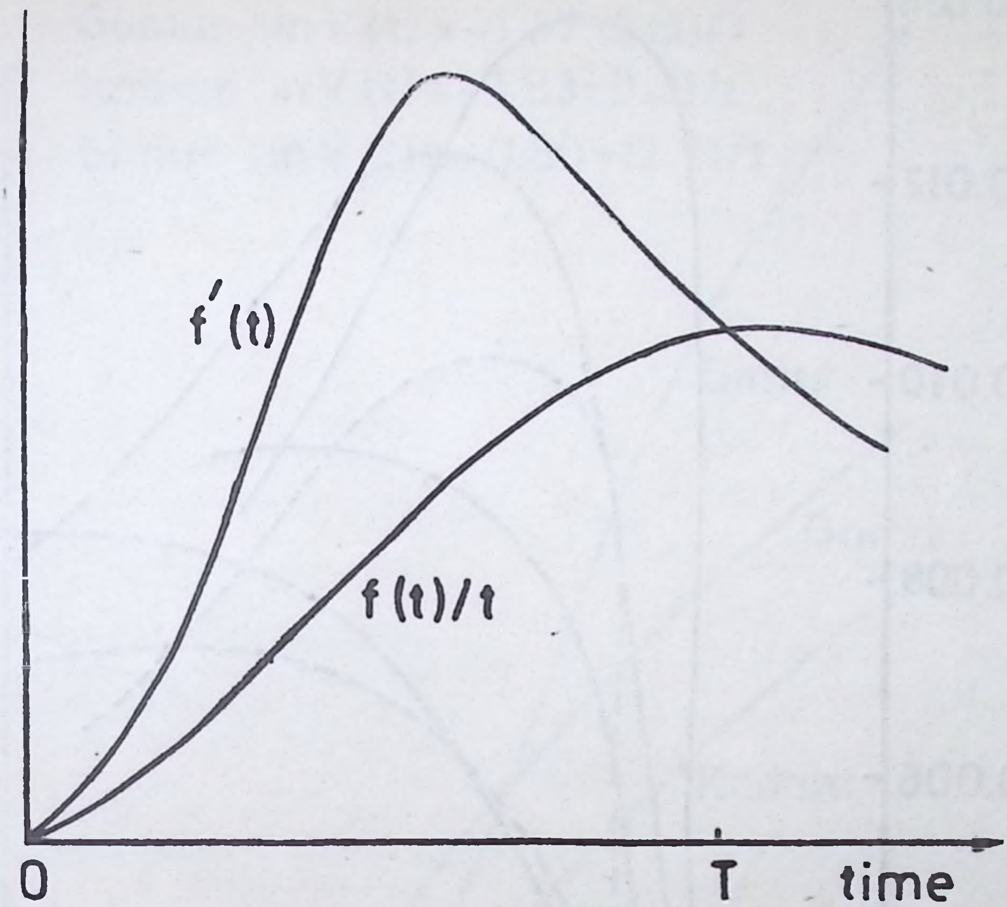

Fig. 1. Qualitative behaviour of the functions $f'(t)$ and $f(t)/t$ against time

Hence, the relationship (18) must be replaced by

$$\ln V(t)=a-T/t,\ t>0. \quad (20)$$

Now, since

$$V''(t)=Te^{a-T/t}(T-2t)/t^4,$$

by Def. 2, $V'(t)$ is convexly increasing on (0, T/2) and is concavely increasing on (T/2, ∞), and so $V'(t)$ has the maximum at T/2. This shows that the condition (15) is satisfied. Finally, the condition (16) is also satisfied as can be seen from the following expression :

$$\frac{d}{dt}\left[V(t)/t\right]=e^{a-T/t}(T-t)/t^3.$$

From the above discussions, it follows that, if V(t) is defined through the equation (20), then all the five conditions (12)–(16) hold true. Hence, by Remark 2, the relationship (20) may be one to express the volume-age relationship.

Remark 3 : Choudhury and Chowdhury (1983) did not give the mathematical reasoning for choosing a particular relationship and just insisted that their model best

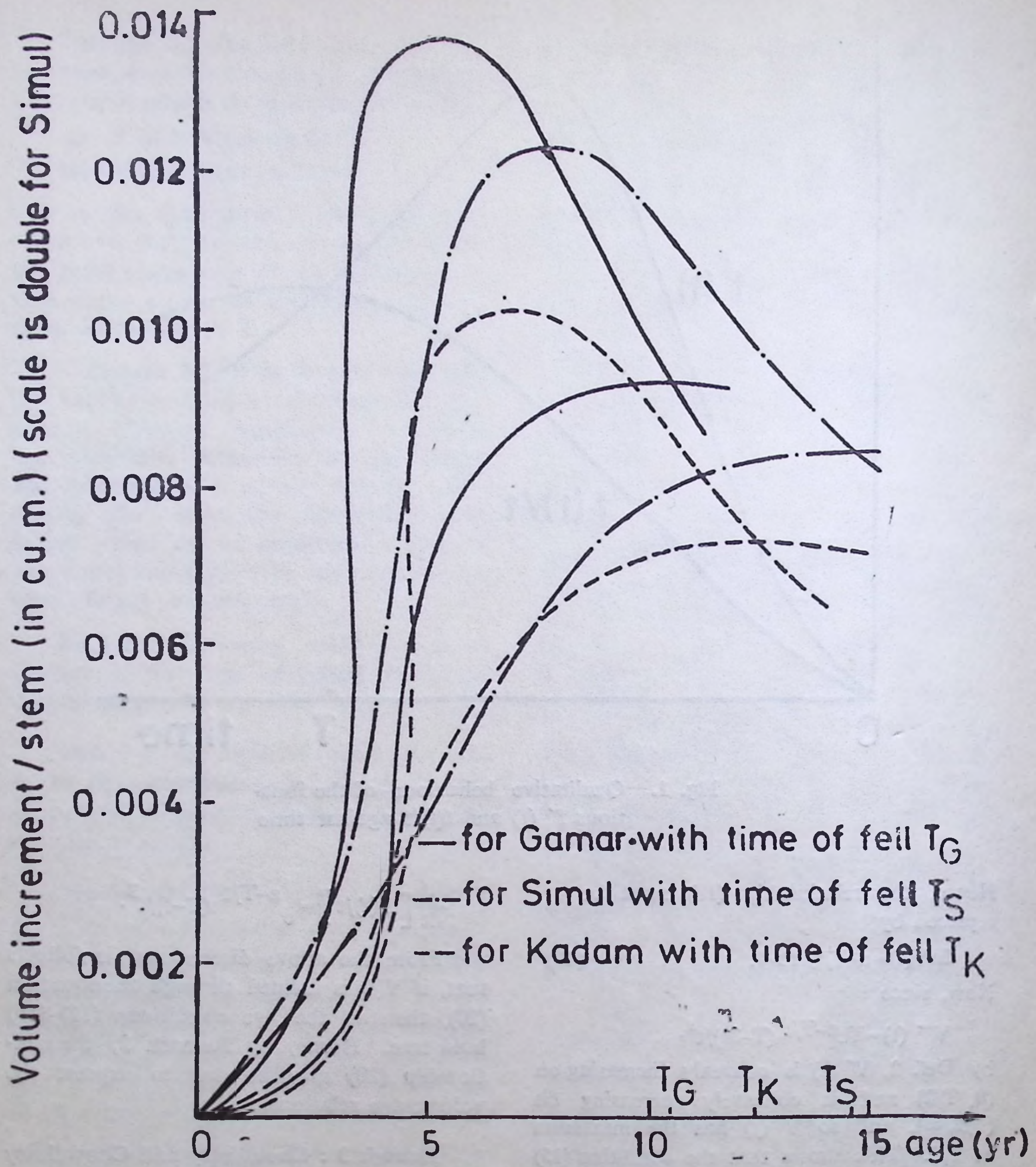

Fig. 2. CAI (upper) and MAI (lower) curves for Gamar, Simul and Kadam
Source ; Choudhury and Chowdhury (1983)

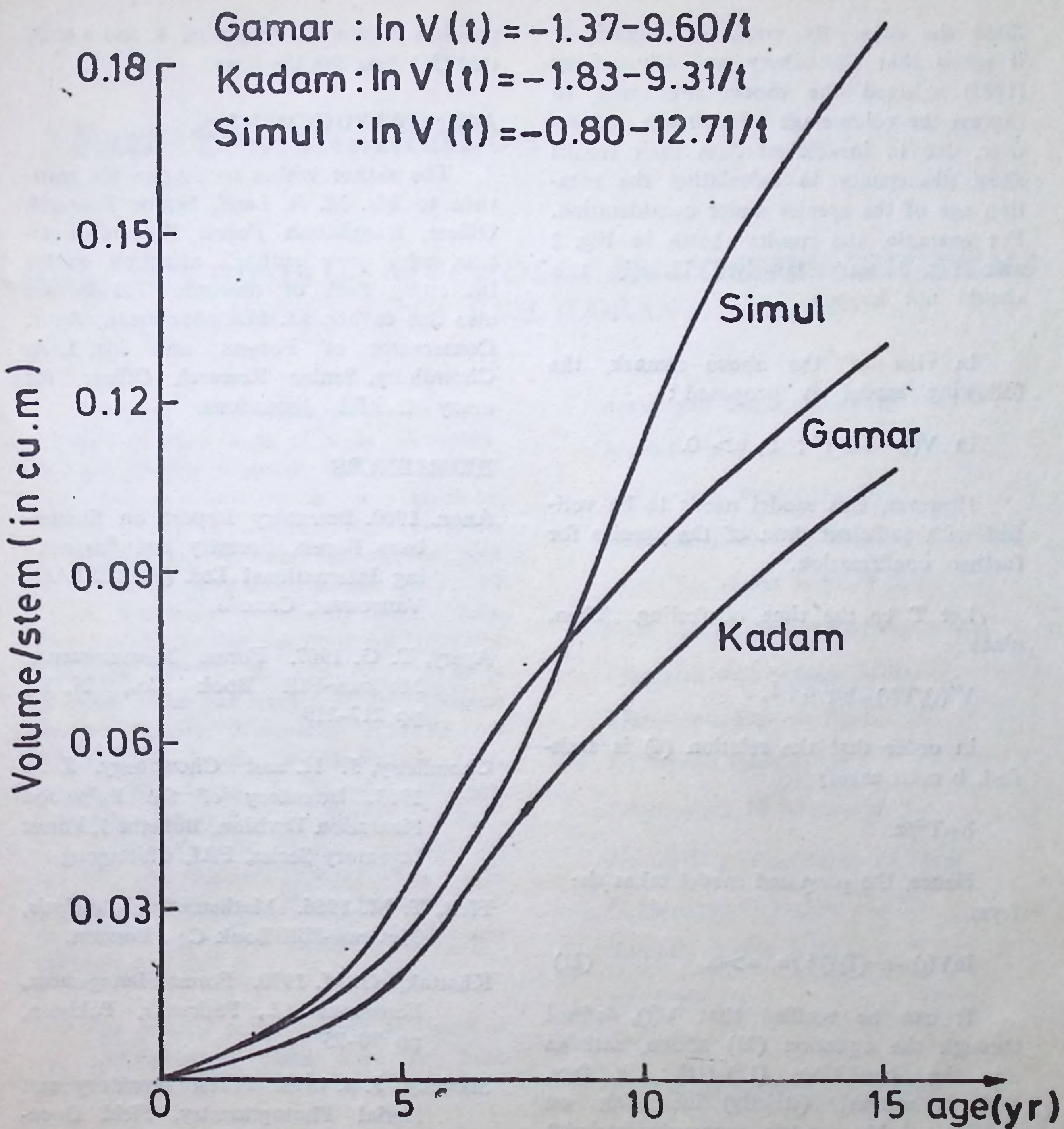

Fig. 3. Age–volume relationship for Gamar, Simul and Kadam
Source : Choudhury and Chowdhury (1983)

fitted the data. By virtue of remark 2, it seems that Choudhury and Chowdhury (1983) selected the model arbitrarily to express the volume-age relationship. Moreover, due to insufficient data their results show discrepancy in calculating the rotation age of the species under consideration. For example, the results shown in Fig. 2 and Fig. 3 vary largely. Usually this should not happen.

In view of the above remark, the following model is proposed :

$$\ln V(t)=a-b/t^{\alpha}\ ;\ b,\ \alpha>0.$$

However, this model needs to be verified with sufficient data of the species for further confirmation.

Let T be the time of feeling. Then, since

$$V'(t)/V(t)=b/t^{\alpha+1},$$

in order that the relation (6) is satisfied, b must satisfy

$$b=T^{\alpha}/\alpha.$$

Hence, the proposed model takes the form

$$\ln V(t)=a-(T/t)^{\alpha}/\alpha,\ \alpha>0. \quad (21)$$

It can be verified that V(t), defined through the equation (21) above, satisfies all the conditions (13)-(16) ; in fact, V(t) is convexly (strictly) increasing on $(0, T/(1+\alpha)^{1/\alpha})$ and is concavely (strictly) increasing on $(T/(1+\alpha)^{1/\alpha}, \infty)$. The problem is then to determine a and α such that (21) best fits the data.

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N. B.—The letter E stands for epsilon